THE JOINT LAW OF THE MAXIMUM AND TERMINAL VALUE OF A MARTINGALE

by

L. C. G. Rogers

Queen Mary & Westfield College, University of London *

ABSTRACT. In this paper, we characterise the possible joint laws of the maximum and terminal value of a uniformly-integrable martingale. We also characterise the joint laws of the maximum and terminal value of a convergent continuous local martingale vanishing at zero. A number of earlier results on the possible laws of the maximum can be deduced quite easily.

Keywords: martingale, terminal value, uniformly integrable, excursion, Brownian motion.

AMS 1991 subject classifications: 60G44 60J65 60G55

* Now at School of Mathematical Sciences, University of Bath, Bath BA2 7AY, Great Britain, email: lcgr@maths.bath.ac.uk. This article appeared in *Probability Theory and Related Fields* **95**, 451–466 (1993).

THE JOINT LAW OF THE MAXIMUM AND TERMINAL VALUE OF A MARTINGALE

by

L. C. G. Rogers

Queen Mary & Westfield College, University of London

1. Introduction. Let $(M_t)_{t\geq 0}$ be a uniformly integrable (UI) martingale and let $\bar{M}_t \equiv \sup\{M_u : u \leq t\}$ be its supremum process. The fact that M is a martingale imposes certain distributional constraints on the laws of M_∞ and $\bar{M}_\infty$, and these have been investigated by a number of authors: see, for example, Blackwell & Dubins [3], Dubins & Gilat [4], Azéma & Yor [2], Perkins [7], Kertz & Rösler [6], Vallois [10]. If ν denotes the law of M_∞, then it was shown by Blackwell & Dubins that

$$\nu \preceq P(\bar{M}_\infty \in \cdot) \preceq \nu^*, \tag{1.1}$$

where $\preceq$ denotes stochastic ordering of probabilities on $\mathbb{R}$ (so that $\nu_1 \preceq \nu_2$ means that $\nu_1((a,\infty)) \leq \nu_2((a,\infty))$ for all $a \in \mathbb{R}$), and ν^* denotes the Hardy transform of ν. [Recall the definition of ν^* in the case where ν has no atoms: if

$$b_\nu(x) \equiv \begin{cases} \int_{[x,\infty)} y\nu(dy)/\nu([x,\infty)) & \text{if } \nu([x,\infty)) > 0; \\ x & \text{if } \nu([x,\infty)) = 0 \end{cases} \tag{1.2}$$

is the barycentre function of ν and if Z has law ν, then ν^* is the law of $b_\nu(Z)$.] Kertz & Rösler went on to prove that if λ is *any* probability on $\mathbb{R}$ satisfying

$$\nu \preceq \lambda \preceq \nu^*, \tag{1.3}$$

where ν has a finite first moment, then there exists a UI martingale such that the law of M_∞ is ν and the law of $\bar{M}_\infty$ is λ. Vallois [10] subsequently characterised those laws λ which could arise if M was assumed also to be continuous.

While these earlier works have discussed only stochastic inequalities between M_∞ and $\bar{M}_\infty$, in this paper, we characterise the possible *joint* law of the variables $(\bar{M}_\infty, M_\infty)$. The characterisation which we obtain is extremely simple, and allows many of the earlier results to be deduced as corollaries. Suppose that μ is the law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$; thus μ is a probability measure on $\mathbb{R}\times\mathbb{R}^+$. If we define

$$c(s) \equiv \begin{cases} \int\int_{(s,\infty)\times\mathbb{R}^+} (x-y)\mu(dx,dy)/\mu((s,\infty)\times\mathbb{R}^+) & \text{if } \mu((s,\infty)\times\mathbb{R}^+) > 0 \\ s & \text{if not,} \end{cases} \tag{1.4}$$

then c has the interpretation

$$c(s) = E(M_\infty | \bar{M}_\infty > s). \tag{1.5}$$

It is easy to see that $c(\cdot)$ must be increasing and $c(s) \geq s$ (Proposition 2.1), so that the conditions

(1.6i) $\int\int |x-y|\mu(dx,dy)(= E|M_\infty|) < \infty$

(1.6ii) *$c(\cdot)$ is increasing;*

(1.6iii) *$c(s) \geq s$ for all s*

are necessary for μ to be the joint law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some UI martingale: the first result of §2 is that *conditions (1.6i–iii) are also sufficient for μ to be the joint law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some UI martingale M* (Theorem 2.2). On the way to proving this, we establish the useful result (Lemma 2.3) that if X is an a.s. convergent continuous local martingale, $X_0 = 0$, then X is a UI martingale if and only if

(1.7i) $E|X_\infty| < \infty$;

(1.7ii) $EX_\infty = 0$;

(1.7iii) $\lim_{a\uparrow\infty} aP(\sup_t X_t > a) = 0$.

If we now restricted attention to UI martingales M such that $M_0 = 0$, the law μ of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ would have to satisfy the further obvious conditions

(1.6iv) *μ is concentrated on $\mathbb{R}^+ \times \mathbb{R}^+$;*

(1.6v) $\int\int (x-y)\mu(dx,dy)(= EM_\infty) = 0$.

The second result of §2 (Corollary 2.4) is that *conditions (1.6i–v) are sufficient for μ to be the law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some UI martingale M vanishing at 0.*

The hard part of the proofs, the sufficiency, requires construction of a martingale, given the law μ satisfying (1.5) and (1.6). The construction used is a variant of the Azéma–Yor [2] embedding (see also Rogers [8]). It magically produces the 'right' martingale, and can be applied to prove the result of Kertz & Rösler in a few lines. The proof makes essential use of excursion theory.

Section 3 of this paper concentrates on the continuous case with $M_0 = 0$. While the problem sounds similar to that of section 2, the methods used are different. The first result characterises the possible laws μ of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$, where M is a convergent continuous local martingale (equivalently, a Brownian motion stopped at a finite stopping time T.) We prove that the condition

$$\mu((t,\infty) \times \mathbb{R}^+)dt \geq \int_{(0,\infty)} y\mu(dt,dy) \tag{1.8}$$

is necessary and sufficient for μ to arise in this way, and that equality holds for a UI martingale. The proof is based on compensating a jump down from the maximum, and is given in Section 3.

Finally in Section 4 we use the earlier methods to characterise the joint laws of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for *uniformly integrable* continuous martingales vanishing at 0, and deduce from this the characterisation due to Vallois [9] of all possible laws of $\bar{M}_\infty$ in that case.

2. The uniformly-integrable case. We begin this section with a simple result of interest in its own right.

PROPOSITION 2.1. *Let M be a uniformly-integrable martingale, and define*

$$c(s) \equiv \begin{cases} E(M_\infty | \bar{M}_\infty > s) & \text{if } P(\bar{M}_\infty > s) > 0 \\ s & \text{if } P(\bar{M}_\infty > s) = 0 \end{cases} \tag{2.1}$$

Then the function $c(\cdot)$ is increasing, and $c(s) \geq s$ for all s.

Proof. We take $x < y$ and prove that $c(x) \leq c(y)$. If $P(\bar{M}_\infty > y) = 0$, then $c(x) \leq y$ and there is nothing left to prove, so we suppose that $P(\bar{M}_\infty > y) > 0$. Now define for each $a \in \mathbb{R}$

$$\tau_a \equiv \inf\{u : M_u > a\}$$

and observe that $\{\bar{M}_\infty > x\} = \{\tau_x < \infty\}$, and

$$c(x) \equiv E(M_\infty | \bar{M}_\infty > x) = E(M(\tau_x) | \tau_x < \infty) \geq x$$

by uniform integrability. The statement that $c(s) \geq s$ is now obvious.

Let $X \leq Y$ be two random variables, $X \in L^1$, such that for all $t \in \mathbb{R}$,

$$E(X|Y > t) \geq t.$$

Then if $x < y$ and we define $A = \{Y > y\}$, $B = \{x < Y \leq y\}$, we have the estimates

$$\begin{aligned} E(X|Y > x) &= \frac{E(X:A) + E(X:B)}{P(A) + P(B)} \\ &\leq \frac{E(X:A) + yP(B)}{P(A) + P(B)} \\ &\leq \frac{E(X:A)}{P(A)}, \end{aligned}$$

since $E(X:A) \geq yP(A)$. Taking $X = M_\infty$ and $Y = \bar{M}_\infty$ yields the desired result. $\square$

The principal result of this section is the following.

THEOREM 2.2. *In order that the probability measure μ on $\mathbb{R} \times \mathbb{R}^+$ should be the law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some UI martingale M, it is necessary and sufficient that*

(2.2i) $\int\int |x - y| \mu(dx, dy) < \infty$;
(2.2ii) $c(\cdot)$ *is increasing;*
(2.2iii) $c(s) \geq s$ *for all* s,

where c is defined in terms of μ by

$$c(s) \equiv \begin{cases} \int\int_{(s,\infty)\times\mathbb{R}^+} (x-y)\mu(dx,dy)/\mu((s,\infty)\times\mathbb{R}^+) & \text{if } \mu((s,\infty)\times\mathbb{R}^+) > 0 \\ s & \text{if not,} \end{cases} \tag{2.3}$$

Proof. The necessity of (2.2i) is obvious, and the necessity of (2.2ii–iii) follows from Proposition 2.1.

For the converse, suppose given (X, Y) with law μ satisfying (2.2i–iii). By shifting the law μ in the x-direction, we may and shall assume that

$$\int\int (x-y)\mu(dx,dy) = 0. \tag{2.4}$$

The proof now consists of three main steps:

Step 1: Reduce the problem to the situation where for some function $v : \mathbb{R} \to \mathbb{R}^+$,

$$y = v(x) \qquad \mu\text{–a.e.};$$

Step 2: Construct the martingale by embedding in a Brownian motion;

Step 3: Confirm that the martingale constructed has the desired properties.

The heart of the proof, Step 2, is a modification of the Azéma-Yor Skorokhod embedding [2].

Proof of Step 1. Let $\mu(dy|x)$ be a regular conditional distribution for Y (to be thought of as $\bar{M}_\infty - M_\infty$) given $X = x$. Let λ be the marginal law of X (to be thought of as $\bar{M}_\infty$), so that

$$\mu(dx, dy) = \lambda(dx)\mu(dy|x). \tag{2.5}$$

Define also

$$\begin{aligned} v(x) &\equiv \int_{\mathbb{R}^+} y\mu(dy|x) \\ &= E[Y|X=x]. \end{aligned} \tag{2.6}$$

This allows us to express c as

$$c(s) = \int_{(s,\infty)} (x - v(x))\lambda(dx)/\bar{\lambda}(s) \tag{2.7}$$

where

$$\bar{\lambda}(s) \equiv \lambda((s,\infty)).$$

We claim that it will be sufficient to construct a UI martingale M with the properties

(2.8i) $\quad \bar{M}_\infty$ *has law* λ;
(2.8ii) $\quad \bar{M}_\infty - M_\infty = v(\bar{M}_\infty)$.

Indeed, suppose that we take such a martingale, and on a suitably enlarged probability space we define a new martingale

$$N_t = \begin{cases} M(t/(1-t)) & (0 \le t < 1) \\ M_\infty & (1 \le t < 2) \\ Z & (2 \le t) \end{cases}$$

where the law of Z given $(M_t)_{t\geq 0}$ is specified by

$$\bar{M}_\infty - Z \sim \mu(\cdot|\bar{M}_\infty).$$

[To see that N *is* a martingale, we need only check

$$\begin{aligned} E[N_2|\tilde{\mathcal{F}}_1] &= E[Z|\tilde{\mathcal{F}}_1] \\ &= \bar{M}_\infty - \int y\mu(dy|\bar{M}_\infty) \\ &= \bar{M}_\infty - v(\bar{M}_\infty) \\ &= M_\infty. \qquad] \end{aligned}$$

Then we claim that $(\bar{N}_\infty, \bar{N}_\infty - N_\infty)$ has law μ. Indeed, $\bar{M}_\infty - Z \geq 0$, so that $\bar{N}_\infty = \bar{M}_\infty$, and

$$\begin{aligned} P(\bar{N}_\infty \in dx, \bar{N}_\infty - N_\infty \in dy) &= P(\bar{M}_\infty \in dx, \bar{M}_\infty - Z \in dy) \\ &= \lambda(dx)\mu(dy|x) \\ &= \mu(dx, dy) \end{aligned}$$

from (2.5). So we may replace $\mu(dx, dy)$ by $\lambda(dx)\delta_{v(x)}(dy)$, and suppose in addition to (2.2) that $y = v(x)$ μ-a.e..

Proof of Step 2. Take a Brownian motion B, $B_0 = 0$, and define

$$S_t \equiv \sup_{u\leq t} B_u,$$

$$T \equiv \inf\{u : B_u \leq h(S_u)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ is the function

(2.9) $$h(s) \equiv c^{-1}(s) - v(c^{-1}(s)).$$

It will turn out that $T < \infty$ a.s. with this choice of h.

To give the construction, we define

(2.10i) $$A_t \equiv \int_0^t I_{\{B_u < c^{-1}(S_u)\}} du,$$

(2.10ii) $$\tau_t \equiv \inf\{u : A_u > t\},$$

(2.10iii) $$M_t \equiv B(\tau_t \wedge T).$$

The process M is a UI martingale with $\bar{M}_\infty \sim \lambda$, $\bar{M}_\infty - M_\infty = v(\bar{M}_\infty)$. The task of Step 3 is to prove this.

[*Explanatory remarks.* (i) Azéma & Yor took $h = b_\nu^{-1}$ and thereby embedded the law ν: $B_T \sim \nu$. However, in view of the definition of the Hardy transform ν^* of ν, we have (at least when ν has no atoms) that for the Azéma-Yor embedding

$$S_T = b_\nu(B_T) \sim \nu^*$$

and the supremum of B^T is stochastically as large as it can be! In order to embed the *given* joint law with a stochastically smaller supremum, we cut out parts of the time axis so that the supremum of the Brownian motion on the part of the time axis which remains is *smaller* than S.

(ii) It is not *a priori* clear that the process M defined by (2.10) *is* a UI martingale. However, we shall prove in Step 3 that $B(\cdot \wedge T)$ is a UI martingale, and it will therefore follow that M is a UI martingale.

(iii) Notice that $\lim_{s\downarrow -\infty} c(s) = 0$ (in view of the assumption (2.4)), c is right-continuous increasing and $c(s) \geq s$ (Proposition 2.1), and most importantly,

$$\bar{M}_\infty = c^{-1}(S(\tau_t \wedge T)). \tag{2.11}$$

Look at Figure 1, which shows a sample path of (B_t, S_t). The lower (triangular-shaped) shaded region, is never entered, and the process stops when it enters the upper (irregularly-shaped) shaded region. The sample path of the process (B, S) consists of lots of horizontal spikes sticking out from the line $B = S$. Some of these are shown; the dotted parts of the lines correspond to parts of the sample path that are excised by the time-change τ, because on these parts of the path, $B > c^{-1}(S)$. Thus the path of the process M as drawn begins suddenly at some negative value, from which it reflects like a Brownian motion for some time before jumping up to a higher value.]

Proof of Step 3. The variables $\bar{M}_\infty$ and M_∞ which concern us are functions of S_T:

$$\bar{M}_\infty = c^{-1}(S_T), \qquad M_\infty = h(S_T). \tag{2.12}$$

Thus from the form (2.9) of h,

$$\bar{M}_\infty - M_\infty = v(\bar{M}_\infty)$$

and so (2.8i) holds. Next we prove (2.8ii). From excursion theory, it is immediate that

$$P(S_T > x) = \exp\Big(-\int_0^x \frac{dt}{t - h(t)}\Big) \tag{2.13}$$

-see Rogers [8] for numerous examples of such calculations. Notice that each side of (2.13) is continuous except possibly at $\sigma \equiv \inf\{x : P(S_T > x) = 0\}$. The elementary implications

$$c^{-1}(S) > y \Rightarrow S \geq c(y), \qquad S > c(y) \Rightarrow c^{-1}(S) \geq y$$

together with (2.13) yield

$$P(\bar{M}_\infty > y) \leq P(S_T \geq c(y)) = P(S_T > c(y)) \leq P(\bar{M}_\infty \geq y)$$

at least for $c(y) \neq \sigma$. Thus we see that *we can achieve (2.8) provided we can pick h so that*

$$\exp\Big(-\int_0^{c(y)} \frac{dt}{t - h(t)}\Big) = \bar{\lambda}(y) \quad \text{for all continuity points } y \text{ of } \lambda; \tag{2.14}$$

In view of the right-continuity of each side of (2.14) it is equivalent to prove that

$$\int_0^{c(y)} \frac{dt}{t - c^{-1}(t) + v(c^{-1}(t))} = -\log \bar{\lambda}(y) \tag{2.15}$$

for $y < \sigma$. Since $c(y) \to 0$ as $y \downarrow -\infty$, each side of (2.15) has the same value at $-\infty$. The two sides of (2.15) jump at the same values of y; if a is some such value, the jump of the left-hand side at a is

$$\begin{aligned}\int_{c(a-)}^{c(a)} \frac{dt}{t - a + v(a)} &= \log\left[\frac{c_a - a + v(a)}{c_{a-} - a + v(a)}\right] \\ &= \log[\bar{\lambda}_{a-}/\bar{\lambda}_a]\end{aligned}$$

as is readily confirmed from the definition (2.3) of c. Finally, writing C for the set of continuity points of λ, the continuous part of the left-hand side of (2.15) is (using (2.6))

$$\begin{aligned}I_C(t)\,\frac{dc_t}{c_t - t + v_t} &= \frac{I_C(t)}{c_t - t + v_t}\left[-\frac{(t - v_t)d\lambda_t}{\bar{\lambda}_t} + \frac{c_t d\lambda_t}{\bar{\lambda}_t}\right] \\ &= I_C(t)\frac{d\lambda_t}{\bar{\lambda}_t}\end{aligned}$$

which is the continuous increasing part of the right-hand side of (2.15). Thus (2.14) holds, so in particular

$$P(S_T > c(y)) = \bar{\lambda}_y \to 0 \text{ as } y \uparrow \infty \tag{2.16}$$

implying that $P(S_T < \infty) = 1 = P(T < \infty)$.

To confirm that the process $M_t \equiv B(\tau_t \wedge T)$ not only satisfies (2.8) but is also a UI martingale, we invoke the following pretty result.

LEMMA 2.3. *Let X be a continuous local martingale, $X_0 = 0$, $\langle X\rangle_\infty < \infty$ a.s.. Then X is a UI martingale if and only if*

(2.17i) $X_\infty \in L^1$;
(2.17ii) $EX_\infty = 0$;
(2.17iii) $\lim_{a\uparrow\infty} aP(\bar{X}_\infty > a) = 0$.

Proof. The necessity of (2.17i–ii) is evident, and the necessity of (2.17iii) follows since $aP(\bar{X}_\infty > a) = E(X_\infty : \bar{X}_\infty > a) \downarrow 0$ as $a \uparrow \infty$.

Next suppose that conditions (2.17) are satisfied, and define

$$H_a \equiv \inf\{t : X_t = a\}, \qquad T_a \equiv H_a \wedge H_{-a}$$

for $a \in \mathbb{R}$. Then

$$0 = EX(T_a) = E[X_\infty : T_a = \infty] + aP[H_a < H_{-a}] - aP[H_{-a} < H_a]; \tag{2.18}$$

from (2.17iii),

$$aP[H_a < H_{-a}] \le aP[H_a < \infty] \le aP[\bar{X}_\infty \ge a] \to 0,$$

and by (2.17i–ii) the first term on the right of (2.18) goes to 0, from which we conclude that

$$aP[H_{-a} < H_a] \to 0$$

as $a \uparrow \infty$. But

$$aP[H_a < H_{-a}] \le aP[H_a < \infty] \to 0,$$

so we deduce that $aP[H_{-a} < \infty] \to 0$, hence that $aP[T_a < \infty] \to 0$. We may now invoke the result of Azéma, Gundy & Yor provided we can check the condition $\sup_t E|X_t| < \infty$. But

$$\begin{aligned} E|X_{t\wedge T_n}| &\le E|X_{T_n}| \\ &= nP[T_n < \infty] + E[|X_\infty| : T_n = \infty] \\ &\le E|X_\infty| + 1 \quad \text{for large } n. \end{aligned}$$

Hence by Fatou's Lemma, $E|X_t| \le E|X_\infty| + 1$, completing the proof. □

We apply this to the continuous local martingale $B(\cdot \wedge T)$, noting that $T < \infty$ a.s.. Since $B_T = h(S_T) = M_\infty$, and $(\bar{M}_\infty, \bar{M}_\infty - M_\infty) \sim \mu$, conditions (2.17i–ii) follow from (2.2i). Next, we have

$$\begin{aligned} c(x)P(S_T > c(x)) &\le c(x)P(c^{-1}(S_T) > x) \\ &= c(x)P(\bar{M}_\infty > x) \\ &= \int_{(x,\infty)} (t - v(t))\lambda(dt) \quad \text{from (2.6)} \\ &\downarrow 0 \quad \text{as } x \uparrow \infty, \end{aligned}$$

verifying condition (2.17iii), at least if c has no jumps. But if for some x one had $c(x-) = a < c(x) = b$, then in the interval $(a, b]$ the functions c^{-1} and h are constant, and so it is easy to see from (2.13) that $zP(S_T > z)$ is increasing throughout the interval $(a, b]$, and so condition (2.17iii) certainly holds. Thus B^T is a UI martingale, from Lemma 2.3, and so therefore is M, by (2.10iii). □

The situation where $M_0 = 0$ now follows easily.

COROLLARY 2.4. *A probability μ on $\mathbb{R}^+ \times \mathbb{R}^+$ is the distribution of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some UI martingale M with $M_0 = 0$ if and only if*

(2.19i) $\int\int |x - y|\mu(dx, dy) < \infty$;
(2.19ii) *$c(\cdot)$ is increasing;*
(2.19iii) *$c(s) \ge s$ for all s;*
(2.19iv) $\int\int (x - y)\mu(dx, dy) = 0$,

where $c(\cdot)$ is defined as before by (2.3).

Proof. The necessity of (2.19i–iv) is evident. Conversely, assuming condition (2.19i–iv), the construction (2.9) used in the proof of Theorem 2.2 yields a UI martingale M with $(\bar{M}_\infty, \bar{M}_\infty - M_\infty) \sim \mu$. $\square$

Let us now see how the result of Kertz & Rösler [6] follows from Theorem 2.2, or, more exactly, from the explicit embedding constructed there.

We use the Azéma-Yor stopping time

$$T \equiv \inf\{t : B_t < b_\nu^{-1}(S_t)\},$$

where b_ν is the barycentre function defined by (1.2). Then it can be shown quite easily (see Azéma & Yor [2]) that the law of S_T is ν^*, and this is also true without the restriction that ν should have no atoms. Now consider some right-continuous increasing $c : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$x^+ \leq c(x) \leq b_\nu(x) \qquad \text{for all } x; \tag{2.20}$$

We carry out the construction (2.9) to build a UI martingale M such that

$$M_\infty = B_T = b_\nu^{-1}(S_T), \qquad \bar{M}_\infty = c^{-1}(S_T).$$

Thus whatever c we choose subject to (2.20), the law of M_∞ is ν; the law of $\bar{M}_\infty$ on the other hand may be any law satisfying the stochastic bounds

$$\nu \preceq \lambda \preceq \nu^*,$$

by suitable choice of c! Taking $c = b_\nu$ gives the lower bound, taking $c(x) = x^+$ gives the upper bound, for then $\bar{M}_\infty = S_T \sim \nu^*$! Kertz & Rösler's results for martingales started at 0 can likewise be easily deduced.

3. The continuous case. In this section, we shall prove the following.

THEOREM 3.1. *The probability measure μ on $\mathbb{R}^+ \times \mathbb{R}^+$ is the joint law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some almost-surely convergent continuous local martingale M which vanishes at 0 if and only if*

$$\left(\int \int_{(t,\infty)\times\mathbb{R}^+} \mu(ds, dy) \right) dt \geq \int_{(0,\infty)} y\mu(dt, dy). \tag{3.1}$$

If M is also uniformly integrable, then the inequality (3.1) holds with equality:

$$\left(\int \int_{(t,\infty)\times\mathbb{R}^+} \mu(ds, dy) \right) dt = \int_{(0,\infty)} y\mu(dt, dy). \tag{3.2}$$

Proof. Firstly, we prove the necessity of the condition (3.1). Without loss of generality, we take the continuous local martingale to be B^T, where B is a Brownian motion, and T is some finite stopping time (see, for example, Theorem IV.34.11 of Rogers &

Williams[9]). Defining as before $H_a \equiv \inf\{t : B_t = a\}$, we shall suppose initially that $T \le T_K \equiv H_K \wedge H_{-K}$, where $K > 0$ is large. In this case, the key observation is that

$$\tilde{B}_t \equiv B(H_t \wedge T) \quad \textit{is an } (\tilde{\mathcal{F}}_t)\textit{- martingale}, \tag{3.3}$$

where $\tilde{\mathcal{F}}_t \equiv \mathcal{F}(H_t)$. Thus if we abbreviate $S \equiv \sup\{B_t : 0 \le t \le T\}$ and define

$$J_t \equiv (S - B_T) I_{[S,\infty)}(t),$$

then

$$\tilde{B}_t = (t \wedge S) - J_t.$$

The process J has a single upward jump of magnitude $Y \equiv S - B_T \ge 0$ at the stopping time S. If ν is the random measure on $\mathbb{R}^+ \times \mathbb{R}^+$ associated with J, that is,

$$\int \phi(s, y)\nu(ds, dy) \equiv \phi(S, Y)$$

and if $\mu \equiv E\nu$ is the law of (S, Y), then for any non-negative $f \in C_K^\infty(\mathbb{R}^+)$ with integral F,

$$0 = E \int_0^\infty f(s) d\tilde{B}_s = E\Big[\int_0^S f(s)ds - \int\int f(s)y\nu(ds, dy)\Big] \tag{3.4}$$

implying that

$$EF(S) = E[Y f(S)]. \tag{3.5}$$

The general stopping time T not constrained by $T \le T_K$ can now be approximated by $T \wedge T_K$; the limiting form of (3.5) is thus

$$EF(S) \ge E[Y f(S)],$$

by Fatou's lemma, with equality if B^T is uniformly integrable. The statements (3.1) and (3.2) follow immediately since f is arbitrary.

Now we turn to the more interesting (constructive) part of the proof, showing the sufficiency of the condition (3.1). So suppose we are given some probability μ on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying (3.1), and set

$$\mu \equiv \mu_0 + \mu_+,$$

where

$$\mu_0 \equiv \mu|_{[0,\infty)\times\{0\}}, \quad \mu_+ \equiv \mu|_{[0,\infty)\times(0,\infty)}.$$

By slightly abusing notation, we shall also consider μ_0 as a measure on $\mathbb{R}^+$. Define

$$\rho(t) \equiv \int\int_{(t,\infty)\times\mathbb{R}^+} \mu(ds, dy) = P(S > t),$$

and let $\mu_+(dy|s)$ be a kernel from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that

$$\int\int \phi(s,y)\mu(ds,dy) = \int ds \int \phi(s,y)\mu_+(dy|s)$$

for any $\phi \in C_K^\infty((0,\infty)\times(0,\infty))$. Such a kernel can be found because of the assumption (3.1). Now we define a Markov kernel $K(\cdot|\cdot) : \mathcal{B}((0,\infty]) \times \mathbb{R}^+ \to [0,1]$ by setting

$$K(A|s) \equiv \frac{\int_A y\mu_+(dy|s)}{P(S>s)} \equiv \frac{\int_A y\mu_+(dy|s)}{\rho(s)} \tag{3.6}$$

for any Borel subset $A \subseteq (0,\infty)$, and to give a Markov kernel,

$$K(\{\infty\}|s) = 1 - K((0,\infty)|s). \tag{3.7}$$

Of course, this definition is meaningless if $\rho(s) \equiv P(S>s) = 0$; in this case we make the arbitrary definition $K(A|s) \equiv I_A(1)$. Notice how the condition (3.1) enters to allow us to define K; it also assures us that the integral in (3.6) is convergent for a.e. s, and so we may suppose that it is convergent for every s.

Now define the increasing function

$$R_t \equiv -\log P(S>0) + \int_{(0,t]} \frac{\mu_0(ds)}{\rho_{s-}} - \sum_{0<u\le t} \{\log \rho_u - \log \rho_{u-} - \frac{\Delta\rho_u}{\rho_{u-}}\}, \tag{3.8}$$

the terms in the sum being easily seen to be nonpositive. This definition only makes sense if $P(S>t) \equiv \rho(t) > 0$; we define $R_t \equiv +\infty$ for t such that $P(S>t)=0$.

We are now ready to describe the construction which will realise the law μ satisfying (3.1) which we were given. Take on some suitable sample space three independent random elements:

(3.9i) a Brownian motion $(B_t)_{t\ge0}$, $B_0 = 0$;

(3.9ii) a $[0,\infty]$-valued random variable V with the law

$$P(V>t) = \exp(-R_t);$$

(3.9iii) a process $\{Z_t : t \ge 0\}$ with values in $(0,\infty]$, where the Z_t are independent of each other, and the law of Z_t is $K(\cdot|t)$.

We define

$$\begin{aligned} T_+ &\equiv \inf\{u : S_u - B_u > Z(S_u)\}, \\ T_0 &\equiv \inf\{u : S_u > V\}, \\ T &\equiv T_0 \wedge T_+, \end{aligned}$$

where $S_u \equiv \sup_{t\le u} B_t$, as you expected.

We shall now prove that the law of $(S_T, S_T - B_T)$ *is the given law* μ. To see this, we have to analyse the excursion process of $Y_t \equiv S_t - B_t$ away from 0. According to Itô [5], the excursion process is a Poisson point process in $\mathbb{R}^+\times U$, where $U \equiv \{\text{continuous} f : \mathbb{R}^+ \to \mathbb{R}^+ \text{such that for some } \zeta>0, f^{-1}((0,\infty)) = (0,\zeta)\}$. The expectation measure is Leb $\times\, n$, where the characteristic measure n can be specified in a variety of ways. The effect of introducing the process Z is to convert the excursion process in $\mathbb{R}^+ \times U$ into an enlarged Poisson process in $\mathbb{R}^+ \times \tilde{U}$, where $\tilde{U} \equiv U \times (0, \infty]$. Think of $(f, h) \in \tilde{U}$ as an excursion together with a height. The excursion will be distributed according to the characteristic measure n, and the height will be independent of the excursion, with law $K(\cdot|l)$, where l is the local time at which the excursion happens. More formally, if A is a Borel subset of U, $a > 0$, then the number of points of the enlarged Poisson process in $[0, t] \times A \times (a, \infty]$ is a Poisson variable with mean

$$\int_{(0,t]} ds\, n(A)\, K((a, \infty]|s).$$

Thus the number of points of the enlarged Poisson process in $C_t \equiv \{(s, (f, h)) : \sup_v f(v) > h, s \leq t\}$ is a Poisson variable with mean

$$\begin{aligned} &\int_{(0,t]} ds \int_{(0,\infty)} K(dy|s)\, n(\{\sup_v f(v) > y\}) \\ &\qquad = \int_{(0,t]} ds \int_{(0,\infty)} K(dy|s)\, y^{-1} \\ &\qquad = \int_{(0,t]} \frac{ds}{\rho(s)} \int_{(0,\infty)} \mu_+(dy|s). \end{aligned}$$

Thus

$$\begin{aligned} P(T_+ > H_t) &= P(C_t = \emptyset) \\ &= \exp\{-\int_{(0,t]} \frac{ds}{\rho(s)} \int_{(0,\infty)} \mu_+(dy|s)\}. \end{aligned}$$

More simply,

$$P(T_0 > H_t) = P(V > t) = \exp(-R_t)$$

from which by independence

(3.10)

$$\begin{aligned} P(T > H_t) &\equiv P(S_T > t) \\ &= \exp\big[-R_t - \int_{(0,t]} \frac{ds}{\rho_{s-}} \int_{(0,\infty)} \mu_+(dy|s)\big] \\ &= \exp\big[\log P(S > 0) + \int_{(0,t]} \frac{d\rho_s}{\rho_{s-}} + \sum_{0<u\leq t} (\log \rho_u - \log \rho_{u-} - \frac{\Delta\rho_u}{\rho_{u-}})\big] \\ &= \rho(t), \end{aligned}$$

using (3.7), the fact that $-d\rho_s = \mu_0(ds) + ds \int_{(0,\infty)} \mu_+(dy|s)$, and Itô's formula. Thus the stopping time T constructed has the property that the law of S_T is $-d\rho$. Next we identify the law of (S_T, Y_T) on the set where $Y_T > 0$. The only way that we can have $Y_T > 0$ is if $T_+ < T_0$, and from the excursion description

$$\begin{aligned} P(T_+ < T_0, S_{T_+} \in ds, Y_{T_+} \in dy) &= e^{-R(s)} \exp\big[-\int_0^s \frac{dv}{\rho_v} \int_{(0,\infty)} \mu_+(dy|v)\big]\, \frac{1}{y}\, K(dy|s) ds \\ &= \rho(s)\, \frac{1}{y}\, \frac{y\mu_+(dy|s)}{\rho(s)} ds, \text{ using (3.6) and (3.10);} \\ &= \mu_+(ds, dy). \end{aligned}$$

Lastly, we must check the law of (S_T, Y_T) on the set where $Y_T = 0$. But the only way $Y_T = 0$ can happen is if $T_0 < T_+$, and

$$P(S(T_0) \in dt, T_0 < T_+) = -d(\exp(-R_t)) \exp\{-\int_0^t \frac{ds}{\rho_s} \int_{(0,\infty)} \mu_+(dy|s)\}.$$

If t is a continuity point of R (equivalently, of ρ or μ_0), then this is easily seen to be $\mu_0(dt)$, as at (3.10). If t is a jump time of R, it is easy to calculate the jump of the right-hand side (again using (3.10)); it is simply $\Delta\mu_0(t)$. Hence

$$P[S_T \in dt, Y_T = 0] = \mu_0(dt).$$

Remarks. (i) The assumption $M_0 = 0$ is not essential to Theorem 3.1; the result still holds without it, but the notation is more untidy, so we have only dealt with the case $M_0 = 0$.
(ii) The condition (3.1) is necessary but not sufficient for M to be UI, as is demonstrated by the example of Brownian motion stopped when it reaches -1.

4. The continuous uniformly-integrable case. Having characterised the possible joint laws of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for a continuous local martingale M ($M_0 = 0$, $\langle M\rangle_\infty < \infty$ a.s.), we now aim to study the laws of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for a *uniformly-integrable* continuous martingale M, $M_0 = 0$. Of course, Lemma 2.3 is the complete answer to the question at one level, but there remains the interesting question 'What are the possible laws of $\bar{M}_\infty$ in this case?' Vallois [10] has answered this question completely by a direct approach using stochastic calculus, and the aim of this section is to prove and interpret his result in the light of the characterisation of the joint laws which we have given already.

The result we shall prove is the following.

THEOREM 4.1. (Vallois) *Suppose that F is a probability measure on $\mathbb{R}^+$, written as $F(dt) = \rho(t)dt + \alpha(dt)$, where α is singular with respect to Lebesgue measure. Then F is the law of the supremum of some continuous UI martingale M, $M_0 = 0$, if and only if the following conditions hold:*

(4.1i) $\rho(t) > 0$ *for all* $t < a \equiv \sup\{u : F(u) < 1\}$;

(4.1ii) $\lim_{t\to\infty} t\bar{F}(t) = 0$, *where* $\bar{F}(t) \equiv 1 - F(t)$;

(4.1iii) $\int_0^\infty t\alpha(dt) + \int_0^\infty dt|t\rho(t) - \bar{F}(t)| < \infty$.

Proof. Suppose that F is the law of $\bar{M}_\infty$ for some continuous UI martingale M vanishing at 0. Then (4.1i) follows from (3.2), (4.1ii) follows from the result of Azéma, Gundy & Yor [1], and (4.1iii) follows because

$$\begin{aligned} E|M_\infty| &= \int\int |s-y|\mu(ds,dy) \\ &= \int_0^\infty s\mu_0(ds) + \int\int |s-y|\mu_+(ds,dy) \\ &\geq \int_0^\infty s\mu_0(ds) + \int_{s=0}^\infty |\int_{y=0}^\infty (s-y)\mu_+(ds,dy)| \\ &= \int_0^\infty s\mu_0(ds) + \int_0^\infty |s\gamma(s) - \bar{F}(s)|ds, \end{aligned}$$

where $\gamma(t)dt \equiv \int_{y=0}^{\infty} \mu_+(dt, dy)$. If now we write $\mu_0(dt) = \alpha(dt) + \phi(t)dt$, so that $\gamma(t) = \rho(t) - \phi(t)$, we have

$$\begin{aligned} \infty &> E|M_\infty| \\ &\geq \int_0^\infty s\alpha(ds) + \int_0^\infty s\phi(s)ds + \int_0^\infty |s\rho(s) - s\phi(s) - \bar{F}(s)|ds \end{aligned}$$

which implies (4.1iii).

For the converse, we suppose given F satisfying (4.1i–iii), and shall exhibit a probability μ on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfying (3.2), the integrability condition

$$\int\int |s-y|\mu(ds, dy) < \infty, \tag{4.2}$$

and

$$\bar{F}(t) = \int\int_{(t,\infty)\times\mathbb{R}^+} \mu(ds, dy). \tag{4.3}$$

From these, it follows that

$$\int\int (s-y)\mu(ds, dy) = 0; \tag{4.4}$$

indeed, using (4.2), (3.2), and (4.1iii), we have

$$\begin{aligned} \int\int (s-y)\mu(ds,dy) &= \lim_{N\to\infty} \int\int_{[0,N]\times\mathbb{R}^+} (s-y)\mu(ds,dy) \\ &= \lim_{N\to\infty} \int_0^N sF(ds) - \int_0^N \bar{F}(s)ds \\ &= \lim_{N\to\infty} -N\bar{F}(N) \\ &= 0. \end{aligned}$$

Thus if μ satisfies (3.2), it is the law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some continuous local martingale M vanishing at 0, by Theorem 3.1; and if it also satisfies (4.2) and (4.3), it satisfies (4.4) and therefore (Lemma 2.3) is the law of $(\bar{M}_\infty, \bar{M}_\infty - M_\infty)$ for some UI martingale M ((2.17iii) being the same as (4.1ii)).

We construct $\mu = \mu_0 + \mu_+$ by defining $\mu_0 = \alpha$, and $\mu_+(dx, dy) \equiv \rho(x)P(dy|x)dx$, where $P(dy|x)$ is a Markov kernel with the properties

$$\bar{F}(x) = \rho(x)\int yP(dy|x) \qquad (x < a), \tag{4.5}$$

and

$$\int\int |x-y|\mu(dx,dy) = \int_0^\infty x\alpha(dx) + \int_0^\infty \rho(x)dx \int_0^\infty |x-y|P(dy|x) < \infty, \tag{4.6}$$

Condition (3.2) then follows immediately from (4.5) and condition (4.3) follows from the definition of μ, so μ has the properties demanded. We shall achieve (4.6) by ensuring that for each x, $P(\cdot|x)$ is concentrated either on $(0, x]$ or on $[x, \infty)$, which makes

$$\begin{aligned}\int_0^\infty |x-y|P(dy|x) &= |\int_0^\infty (x-y)P(dy|x)| \\ &= |x - \bar{F}(x)/\rho(x)|\end{aligned}$$

using (4.5). Thus

$$\int_0^\infty \rho(x)dx \int_0^\infty |x-y|P(dy|x) = \int_0^\infty |x\rho(x) - \bar{F}(x)|dx < \infty,$$

by assumption (4.1iii). The choice of $P(\cdot|x)$ to ensure (4.5) and (4.6) is easy to make; if $\bar{F}(x)/\rho(x) \le x$, then we take $P(\cdot|x)$ to be concentrated on $[0, x]$ in such a way that

$$\int yP(dy|x) = \frac{\bar{F}(x)}{\rho(x)},$$

and similarly if $\bar{F}(x)/\rho(x) > x$.

Acknowledgement. It is a pleasure to thank two thorough referees for their detailed and perceptive reading of the paper, which resulted in many improvements.

REFERENCES

[1] AZEMA, J., GUNDY, R. F., & YOR, M. (1980). Sur l'intégrabilité uniforme des martingales continues. *Séminaire de Probabilit'es XIV, Lecture Notes in Math.* **784**, 53–61, Springer, Berlin.

[2] AZEMA, J., & YOR, M. (1979). Une solution simple au problème de Skorokhod. *Séminaire de Probabilités XII, Lecture Notes in Math.* **721**, 90–115, Springer, Berlin.

[3] BLACKWELL, D. & DUBINS, L. E. (1963). A converse to the dominated convergence theorem. *Ill. J. Math.* **7**, 508–514.

[4] DUBINS, L. E. & GILAT, D. (1978). On the distribution of maxima of martingales. *Proc. Amer. Math. Soc.* **68**, 337–338.

[5] ITÔ, K. (1971).Poisson point processes attached to Markov processes. *Proc. 6th Berkeley Symp. Math. Statis. Prob., Vol. 3.* Univ. California Press, Berkeley.

[6] KERTZ, R. P. & RÖSLER, U. (1990). Martingales with given maxima and terminal distributions. *Isr. J. Math.* **69**, 173–192.

[7] PERKINS, E. (1986). The Cereteli-Davis solution to the H^1-embedding problem and an optimal embedding in Brownian motion. *Seminar on Stochastic Processes, 1985*, 172–223, Birkhäuser, Boston.

[8] ROGERS, L. C. G. (1981). Williams' characterization of the Brownian excursion law: proof and applications. *Séminaire de Probabilités XV, Lecture Notes in Math.* **850**, 227–250, Springer, Berlin.

[9] ROGERS, L. C. G. & WILLIAMS, D. (1987). *Diffusions, Markov Processes, and Martingales, Vol. 2.* Wiley, Chichester.

[10] VALLOIS, P. (1992). Sur la loi du maximum et du temps local d'une martingale continue uniformément intégrable. *Preprint.*

School of Mathematical Sciences
Queen Mary & Westfield College
Mile End Rd
London E1 4NS
England